Symmetries in the fourth Painlevé equation and Okamoto polynomials

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It is known by K. Okamoto [7] that the fourth Painlevé equation has symmetries under the affine Weyl group of type $A_2^{(1)}$. In this paper we propose a new representation of the fourth Painlevé equation in which the $A_2^{(1)}$-symmetries become clearly visible. By means of this representation, we clarify the internal relation between the fourth Painlevé equation and the modified KP hierarchy. We obtain in particular a complete description of the rational solutions of the fourth Painlevé equation in terms of Schur functions. This implies that the so-called *Okamoto polynomials*, which arise from the τ-functions for rational solutions, are in fact expressible by the 3-reduced Schur functions. [1]

1. A symmetric form of the fourth Painlevé equation

The fourth Painlevé equation $\mathrm{P_{IV}}$ is the following second order ordinary differential equation

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - a)y + \frac{b}{y} \tag{1.1}$$

for the unknown function $y = y(t)$, where $' = d/dt$ and $a, b \in \mathbb{C}$ are parameters. It is known by K. Okamoto [7] that equation (1.1) is represented as the following system for the two unknown functions $q = y$ and p:

$$\begin{aligned} q' &= q(2p - q - 2t) - 2(v_1 - v_2), \\ p' &= p(2q - p + 2t) + 2(v_2 - v_3). \end{aligned} \tag{1.2}$$

This equation, called $\mathrm{H_{IV}}$, is in fact a Hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q} \tag{1.3}$$

with polynomial Hamiltonian

$$H = qp^2 - q^2p - 2tpq - 2(v_1 - v_2)p - 2(v_2 - v_3)q. \tag{1.4}$$

The parameters $\mathbf{v} = (v_1, v_2, v_3)$, $(v_1 + v_2 + v_3 = 0)$ in (1.2) and (a, b) in (1.1) are related through the formulas

$$a = 1 + 3v_3, \quad b = -2(v_1 - v_2)^2. \tag{1.5}$$

The equivalence between (1.1) and (1.2) can be checked directly, but it requires a tedious calculation. (This calculation is fairly simplified by the "symmetric" representation which we will propose in this paper. See the proof of Theorem 1.1 below.)

[1] After completing this paper, the authors were informed by K. Kajiwara and Y. Ohta that they obtained independently the expression of Okamato polynomials in terms of Schur functions.

It is clearly seen from (1.2) that, if $v_1 - v_2 = 0$ or $v_2 - v_3 = 0$, the Hamiltonian system $\mathrm{H_{IV}}$ has *classical solutions* such that $q = 0$ or $p = 0$. In these cases, equation (1.2) is reduced to the Riccati equations $p' = -p^2 + 2tp + 2(v_2 - v_3)$ and $q' = -q^2 - 2t - 2(v_1 - v_2)$ respectively, and they are furthermore linearized to Hermite-Weber equations. In this sense, the Hamiltonian system $\mathrm{H_{IV}}$ (1.2) has *invariant divisors* $q = 0$ and $p = 0$ along the lines $v_1 - v_2 = 0$ and $v_2 - v_3 = 0$, respectively. It should be noted that equation (1.2) has one more typical invariant divisor $q - p + 2t = 0$ along the line $v_1 - v_3 = 1$. In fact equation (1.2) implies

$$(q - p + 2t)' = -(q - p - 2t)(q + p) + 2(1 - v_1 + v_3). \tag{1.6}$$

It is known by [6] that these three polynomials q, p and $q-p+2t$ generate essentially all the invariant divisors of the fourth Painlevé equation (1.2). Note that the three simple affine roots $1 - v_1 + v_3$, $v_1 - v_2$, $v_2 - v_3$ of type $A_2^{(1)}$ are already involved in these equations. We denote by

$$V = \{\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{C}^3 \ ; \ v_1 + v_2 + v_3 = 0\} \tag{1.7}$$

the parameter space for the system (1.2).

We now propose to treat the three typical invariant divisors q, p and $q - p + 2t$ equally so as to obtain a "symmetric" representation of the fourth Painlevé equation. We introduce the three dependent variables $f = (f_0, f_1, f_2)$ as follows. Fixing a nonzero complex number $c \in \mathbb{C}^\times$, set

$$f_0 = c(q - p + 2t), \quad f_1 = -cq, \quad f_2 = cp, \tag{1.8}$$

and rescale the independent variable as $x = -t/c$. Then we have

$$\begin{aligned} f_0' &= f_0(f_2 - f_1) - 2c^2(1 - v_1 + v_3), \\ f_1' &= f_1(f_0 - f_2) - 2c^2(v_1 - v_2), \\ f_2' &= f_2(f_1 - f_0) - 2c^2(v_2 - v_3), \end{aligned} \tag{1.9}$$

where $' = d/dx$. With the normalization $c = \sqrt{-3/2}$, we set

$$\alpha_0 = 3(1 - v_1 + v_3), \quad \alpha_1 = 3(v_1 - v_2), \quad \alpha_2 = 3(v_2 - v_3). \tag{1.10}$$

Then we have

Theorem 1.1. *The fourth Painlevé equation* (1.1) (*or* (1.2)) *can be written in the following symmetric form:*

$$\begin{aligned} f_0' + f_0(f_1 - f_2) &= \alpha_0, \\ f_1' + f_1(f_2 - f_0) &= \alpha_1, \\ f_2' + f_2(f_0 - f_1) &= \alpha_2, \end{aligned} \tag{1.11}$$

with normalization

$$f_0 + f_1 + f_2 = 3x, \tag{1.12}$$

where $' = d/dx$ *and* $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ *are parameters with* $\alpha_0 + \alpha_1 + \alpha_2 = 3$.

Proof. The equation (1.11) has been derived from the Hamiltonian system (1.2); it is clear that these two are equivalent. We will show the equivalence of (1.1) and (1.11) (with normalization (1.12)). This gives in fact an easier way to establish the equivalence between (1.1) and (1.2). Taking a derivative of the second equation of (1.11), we have

$$f_1'' + f_1'(f_2 - f_0) + f_1(f_2' - f_0') = 0.$$

Substituting the first and the third equations of (1.11) to this, we obtain

$$f_1'' + f_1'(f_2 - f_0) - 2f_0f_2f_1 + (\alpha_0 - \alpha_2)f_1 + (f_2 + f_0)f_1^2 = 0.$$

Then, by using the relations

$$f_2 - f_0 = \frac{\alpha_1 - f_1'}{f_1}, \quad f_2 + f_0 = 3x - f_1, \quad 4f_0f_2 = (f_2 + f_0)^2 - (f_2 - f_0)^2,$$

we have

$$f_1'' - \frac{1}{2}\frac{{f_1'}^2}{f_1} - \frac{3}{2}f_1^3 + 6xf_1^2 + \left(-\frac{9}{2}x^2 + (\alpha_0 - \alpha_2)\right)f_1 + \frac{\alpha_1^2}{2}\frac{1}{f_1} = 0.$$

This is transformed into the equation (1.1) by the rescaling $f_1 = -cy$, $x = -t/c$, $c = \sqrt{-3/2}$ and the change of parameters (1.5), (1.10). □

We remark that our equation (1.11) has the following rational solutions:

(1.13)
$$\begin{aligned} &\text{(A)} \qquad (\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (1, 1, 1;\, x, x, x), \\ &\text{(B)} \qquad (\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2) = (3, 0, 0; 3x, 0, 0). \end{aligned}$$

From the work of Y. Murata [4], it follows that all the rational solutions of (1.11) are obtained from these two particular solutions by Bäcklund transformations. There are classical solutions obtained as Bäcklund transformations from the solutions of Riccati type along the three lines $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$. Any other solutions are *non-classical* in the sense of H. Umemura [9] (see also [6], [7]). It should also be noted that our equation (1.9) reduces to the Kac-Moerbeke integrable system [3] in the degenerate limit $c \to 0$.

2. Bäcklund transformations and the affine Weyl group

We now discuss symmetries in the fourth Painlevé equation represented by (1.11) with the normalization of (1.12). In what follows, we regard α_0, α_1, α_2 as coordinate functions (with $\alpha_0 + \alpha_1 + \alpha_2 = 3$) of the parameter space V.

We consider the affine Weyl group $W = \langle s_0, s_1, s_2 \rangle$ of type $A_2^{(1)}$ with fundamental relations

(2.1)
$$s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad (i = 0, 1, 2).$$

Here the subscripts are understood as elements of $\mathbb{Z}/3\mathbb{Z}$. This convention for subscripts will be applied to other variables α_i, f_i, etc., as well. We denote by $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$ the extension of W obtained by adjoining the following Dynkin diagram automorphism π:

(2.2)
$$\pi^3 = 1, \quad \pi s_i = s_{i+1}\pi \quad (i = 0, 1, 2).$$

The affine Weyl group $\widetilde{W}$ acts naturally on the coordinate ring $\mathbb{C}[\alpha]$ of V through the algebra automorphism s_0, s_1, s_2 and π of $\mathbb{C}[\alpha]$ determined by

$$s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \quad (i \neq j), \quad \pi(\alpha_j) = \alpha_{j+1} \tag{2.3}$$

for $i, j = 0, 1, 2$. When we consider the action of $\widetilde{W}$ on the parameter space V, we will use the action such that $(w.\varphi)(\mathbf{v}) = \varphi(w^{-1}.\mathbf{v})$ for any $\mathbf{v} \in V$ and $\varphi \in \mathbb{C}[\alpha]$. The action of $\widetilde{W}$ on V is given as follows:

$$\begin{aligned} s_0.\mathbf{v} &= (v_3 + 1, v_2, v_1 - 1), \quad s_1.\mathbf{v} = (v_2, v_1, v_3), \\ s_2.\mathbf{v} &= (v_1, v_3, v_2), \quad \pi.\mathbf{v} = (v_3 + \frac{2}{3}, v_1 - \frac{1}{3}, v_2 - \frac{1}{3}). \end{aligned} \tag{2.4}$$

for any $\mathbf{v} = (v_1, v_2, v_3) \in V$.

One advantage of our representation (1.11) is that the action of the affine Weyl group $\widetilde{W}$ on the fourth Painlevé equation can be described in a completely symmetric way on the dependent variables f_0, f_1 and f_2. The action of $\widetilde{W}$ on $\mathbb{C}[\alpha]$ extends in fact to the whole differential field $K = \mathbb{C}(\alpha; f)$ as follows.

Theorem 2.1. *The fourth Painlevé equation* (1.11) *is invariant under the following transformations* s_0, s_1, s_2 *and* π*:*

(2.5)
$$\begin{array}{llll} s_0(f_0) = f_0, & s_1(f_1) = f_1, & s_2(f_2) = f_2, & \pi(f_0) = f_1, \\ s_0(f_1) = f_1 - \dfrac{\alpha_0}{f_0}, & s_1(f_2) = f_2 - \dfrac{\alpha_1}{f_1}, & s_2(f_0) = f_0 - \dfrac{\alpha_2}{f_2}, & \pi(f_1) = f_2, \\ s_0(f_2) = f_2 + \dfrac{\alpha_0}{f_0}, & s_1(f_0) = f_0 + \dfrac{\alpha_1}{f_1}, & s_2(f_1) = f_1 + \dfrac{\alpha_2}{f_2}, & \pi(f_2) = f_0, \\ s_0(\alpha_0) = -\alpha_0, & s_1(\alpha_1) = -\alpha_1, & s_2(\alpha_2) = -\alpha_2, & \pi(\alpha_0) = \alpha_1, \\ s_0(\alpha_1) = \alpha_1 + \alpha_0, & s_1(\alpha_2) = \alpha_2 + \alpha_1, & s_2(\alpha_0) = \alpha_0 + \alpha_2, & \pi(\alpha_1) = \alpha_2, \\ s_0(\alpha_2) = \alpha_2 + \alpha_0, & s_1(\alpha_0) = \alpha_0 + \alpha_1, & s_2(\alpha_1) = \alpha_1 + \alpha_2, & \pi(\alpha_2) = \alpha_0. \end{array}$$

Furthermore, these transformations define a representation of the affine Weyl group $\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$*. Namely,* $\widetilde{W}$ *acts on the differential field* $K = \mathbb{C}(\alpha; f)$ *as a group of differential automorphisms.*

Theorem 2.1 is proved by straightforward computations. The transformations described above will be called the *Bäcklund transformations* of the fourth Painlevé equation (1.11). Note that the independent variable $x = (f_0 + f_1 + f_2)/3$ is fixed under the action of $\widetilde{W}$.

Note that, for any $w \in W$, one obtains three linear functions $\beta_0 = w(\alpha_0)$, $\beta_1 = w(\alpha_1)$, $\beta_2 = w(\alpha_2)$ in $\alpha_0, \alpha_1, \alpha_2$. Theorem 2.1 then implies that, one can specify certain rational functions $g_0 = w(f_0), g_1 = w(f_1), g_2 = w(f_2)$ in $f_0, f_1, f_2, \alpha_0, \alpha_1, \alpha_2$ such that

$$g_i' + g_i(g_{i+1} - g_{i+2}) = \beta_i \quad (i = 0, 1, 2). \tag{2.6}$$

Namely, if (f_0, f_1, f_2) is a (generic) solution of (1.11) with parameters $(\alpha_0, \alpha_1, \alpha_2)$, then (g_0, g_1, g_2) is again a solution of the same system with parameters $(\beta_0, \beta_1, \beta_2)$.

We give an example below to show how the dependent variables f_0, f_1, f_2 are transformed under the action of the affine Weyl group.

Example. For $w = s_1 s_0$, the Bäcklund transformation $w(f_1) = s_1 s_0(f_1)$ is computed as follows:

$$f_1 \xrightarrow{s_0 \cdot} \frac{f_0 f_1 - \alpha_0}{f_0} \xrightarrow{s_1 \cdot} \frac{f_1(f_0 f_1 - \alpha_0)}{f_0 f_1 + \alpha_1}. \tag{2.7}$$

Similarly we have

$$\begin{aligned} &\beta_0 = w(\alpha_0) = \alpha_2 - 3, \quad \beta_1 = w(\alpha_1) = \alpha_0, \quad \beta_2 = w(\alpha_2) = \alpha_1 + 3; \\ &g_0 = w(f_0) = \frac{f_0 f_1 + \alpha_1}{f_1}, \quad g_1 = w(f_1) = \frac{f_1(f_0 f_1 - \alpha_0)}{f_0 f_1 + \alpha_1}, \\ &g_2 = w(f_2) = \frac{(f_0 f_1 + \alpha_1)(f_1 f_2 - \alpha_1) + (3 - \alpha_2) f_1^2}{f_1(f_0 f_1 + \alpha_1)}. \end{aligned} \tag{2.8}$$

If we specialize these formula to the particular solution

$$(\alpha_0, \alpha_1, \alpha_2\, ;\, f_0, f_1, f_2) = (1, 1, 1;\, x, x, x), \tag{2.9}$$

we obtain another rational solution

$$(\alpha_0, \alpha_1, \alpha_2\, ;\, f_0, f_1, f_2) = (-2, 1, 4\, ;\, \frac{x^2+1}{x}, \frac{x(x^2-1)}{(x^2+1)}, \frac{x^4+2x^2-1}{x(x^2+1)}). \tag{2.10}$$

A complete description of rational functions in x arising in this way will be given later in this paper.

Remark. The Bäcklund transformation $s_0(f_1) = f_1 - \frac{\alpha_0}{f_0}$, for example, becomes singular when applied to a particular solution such that $f_0 = 0$. This sort of problem, however, is only apparent since such a solution arises only under the condition $\alpha_0 = 0$ as one sees immediately from (1.11). When $\alpha_0 = 0$, it is natural to understand that the Bäcklund transformation s_0 becomes the identity transformation. In general, each $g_i = w(f_i)$ is a rational function in $(\alpha; f)$ and its denominator possibly becomes identically zero when one specializes $(\alpha; f)$ to certain particular solutions. Such a phenomenon occurs however only when some of the parameters α_0, α_1, α_2 are in $3\mathbb{Z}$. In such cases, critical factors in the denominator of $g_i = w(f_i)$ can actually be eliminated by specializing the parameters $(\alpha_0, \alpha_1, \alpha_2)$ in advance. With this *regularization*, our Bäcklund transformations $w(f_i)$ make sense for any particular solution.

3. τ-Functions

In this section, we show that our equation (1.11) for $f = (f_0, f_1, f_2)$ can be bilinearized by introducing a triple of τ-functions $\tau = (\tau_0, \tau_1, \tau_2)$. We also study the Bäcklund transformations on the level of τ-functions.

We introduce the *τ-functions* τ_0, τ_1, τ_2 to be the dependent variables satisfying the following equations:

$$\begin{aligned} f_0 &= (\log \frac{\tau_1}{\tau_2})' + x = \frac{\tau_1'}{\tau_1} - \frac{\tau_2'}{\tau_2} + x, \\ f_1 &= (\log \frac{\tau_2}{\tau_0})' + x = \frac{\tau_2'}{\tau_2} - \frac{\tau_0'}{\tau_0} + x, \\ f_2 &= (\log \frac{\tau_0}{\tau_1})' + x = \frac{\tau_0'}{\tau_0} - \frac{\tau_1'}{\tau_1} + x. \end{aligned} \tag{3.1}$$

We fix the freedom of overall multiplication by a function in defining τ_0, τ_1, τ_2, by imposing the equation

$$2(\log \tau_0\tau_1\tau_2)'' + (f_0 - x)^2 + (f_1 - x)^2 + (f_2 - x)^2 = 0. \tag{3.2}$$

To be more precise, we first introduce a variable g (determined from f_0, f_1, f_2 up to an additive constant) as an integral of the equation

$$2g' + (f_0 - x)^2 + (f_1 - x)^2 + (f_2 - x)^2 = 0. \tag{3.3}$$

Then we require that the τ-functions τ_0, τ_1, τ_2 should satisfy

$$g = (\log \tau_0\tau_1\tau_2)' = \frac{\tau_0'}{\tau_0} + \frac{\tau_1'}{\tau_1} + \frac{\tau_1'}{\tau_1}. \tag{3.4}$$

Note that, under the conditions (3.1) and (3.4), the τ-functions τ_0,τ_1,τ_2 are determined by the equations

$$\begin{aligned} (\log \tau_0)' &= \frac{\tau_0'}{\tau_0} = \frac{1}{3}(g - f_1 + f_2), \\ (\log \tau_1)' &= \frac{\tau_1'}{\tau_1} = \frac{1}{3}(g - f_2 + f_0), \\ (\log \tau_2)' &= \frac{\tau_2'}{\tau_2} = \frac{1}{3}(g - f_0 + f_1), \end{aligned} \tag{3.5}$$

up to multiplicative constants, respectively. We remark that the integration constant in g has the effect of multiplying each τ_i by the exponential of a linear function in x.

In order to describe the differential equations to be satisfied by the τ-functions, we recall the definition of Hirota's bilinear equations. Let $P(\partial_x)$ ($\partial_x = d/dx$) be a linear differential operator in the x-variable with constant coefficients. Then Hirota's bilinear operator $P(D_x)$ is defined by

$$P(D_x)\ F(x) \cdot G(x) = P(\partial_y)F(x + y)G(x - y)|_{y=0}, \tag{3.6}$$

for a given pair of functions $F(x)$, $G(x)$.

Theorem 3.1. *The fourth Painlevé equation* (1.11) *for f_0, f_1, f_2, together with the integral g of* (3.3)*, is equivalent to the following system of Hirota bilinear equations*

for the triple of τ-functions τ_0,τ_1,τ_2:

$$
\begin{aligned}
&(D_x^2 - xD_x - \frac{\alpha_0-\alpha_1}{3})\ \tau_0\cdot\tau_1 = 0,\\
&(D_x^2 - xD_x - \frac{\alpha_1-\alpha_2}{3})\ \tau_1\cdot\tau_2 = 0,\\
&(D_x^2 - xD_x - \frac{\alpha_2-\alpha_0}{3})\ \tau_2\cdot\tau_0 = 0.
\end{aligned}
\tag{3.7}
$$

Proof. Note first that, in terms of the logarithms $F_i = \log\tau_i$ $(i = 0, 1, 2)$ of τ-functions, the dependent variables f_0, f_1, f_2 are expressed as follows:

$$
\begin{aligned}
&f_0 = F_1' - F_2' + x, \quad f_1 = F_2' - F_0' + x, \quad f_2 = F_0' - F_1' + x,\\
&g = F_0' + F_1' + F_2'.
\end{aligned}
\tag{3.8}
$$

The three equations of Theorem are rewritten into the following equations for F_0, F_1, F_2:

$$
\begin{aligned}
F_0'' + F_1'' + (F_0' - F_1')^2 - x(F_0' - F_1') - \frac{\alpha_0-\alpha_1}{3} &= 0,\\
F_1'' + F_2'' + (F_1' - F_2')^2 - x(F_1' - F_2') - \frac{\alpha_1-\alpha_2}{3} &= 0,\\
F_2'' + F_0'' + (F_2' - F_0')^2 - x(F_2' - F_0') - \frac{\alpha_2-\alpha_0}{3} &= 0.
\end{aligned}
\tag{3.9}
$$

Taking the sum of these three equations, we have

$$
2(F_0'' + F_1'' + F_2'') + (F_1' - F_2')^2 + (F_2' - F_0')^2 + (F_0' - F_1')^2 = 0,
\tag{3.10}
$$

which corresponds to the equation (3.3) for g. By subtracting the third equation of (3.9) from the first, we have

$$
F_1'' - F_2'' - (F_1' - F_2' + x)(2F_0' - F_1' - F_2') - \alpha_0 + 1 = 0,
\tag{3.11}
$$

which corresponds to the differential equation for f_0. Similarly we have the equations for f_1 and f_2 from (3.9). It is also clear that the equations (3.9) are recovered from (3.10) and the three equations which correspond to (1.11). □

Remark. Consider the differential field $K(g) = \mathbb{C}(\alpha; f)(g)$ obtained from $K = \mathbb{C}(\alpha; f)$ by adjoining a variable g on which the derivation $'$ acts by the formula (3.3). Then Theorem 3.1 implies that this differential field is isomorphic to the differential field $\mathbb{C}(\alpha)(x, F_0', F_1', F_2')$ defined by the relations (3.9). Note that, by (3.9) and (3.10), each second derivative F_i'' $(i = 0, 1, 2)$ can be expressed in terms of x and F_0', F_1', F_2':

$$
F_i'' + x(F_{i+1}' - F_{i+2}') + (F_i' - F_{i+1}')(F_i' - F_{i+2}') + \frac{\alpha_{i+1}-\alpha_{i+2}}{3} = 0
\tag{3.12}
$$

for $i = 0, 1, 2$. This system is also equivalent to the equation (3.7) for the triple τ_0, τ_1, τ_2 of τ-functions. Note that the differential field of our τ-functions is defined as $\mathbb{C}(\alpha)(x, \tau_0, \tau_1, \tau_2, \tau_0', \tau_1', \tau_2')$ by (3.12), regarded as equations for τ-functions.

One important fact is that the action of the affine Weyl group on the f-variables lifts to the level of τ-functions.

Theorem 3.2. *The τ-functions (τ_0, τ_1, τ_2) allow an action of the affine Weyl group $\widetilde{W}$ which is compatible with the action of $\widetilde{W}$ on f_0, f_1, f_2 of Theorem* 2.1. *Their Bäcklund transformations are again expressed by Hirota's bilinear operators as follows:*

$$
\begin{aligned}
s_0(\tau_0) &= \frac{1}{\tau_0}(D_x + x)\,\tau_1 \cdot \tau_2 = \frac{1}{\tau_0}(\tau_1'\tau_2 - \tau_1\tau_2' + x\tau_1\tau_2),\\
s_1(\tau_1) &= \frac{1}{\tau_1}(D_x + x)\,\tau_2 \cdot \tau_0 = \frac{1}{\tau_1}(\tau_2'\tau_0 - \tau_2\tau_0' + x\tau_2\tau_0),\\
s_2(\tau_2) &= \frac{1}{\tau_2}(D_x + x)\,\tau_0 \cdot \tau_1 = \frac{1}{\tau_2}(\tau_0'\tau_1 - \tau_0\tau_1' + x\tau_0\tau_1),\\
s_i(\tau_j) &= \tau_j \quad (i \neq j), \quad \pi(\tau_j) = \tau_{j+1} \quad (i, j = 0, 1, 2),
\end{aligned}
\tag{3.13}
$$

while s_0, s_1, s_2 and π act on α_0,α_1,α_2 in the same way as in Theorem 2.1.

Proof. We first extend the action of $\widetilde{W}$ on $\mathbb{C}(\alpha; f)$ to $\mathbb{C}(\alpha; f)(g)$, or equivalently to $\mathbb{C}(\alpha)(x, F_0', F_1', F_2')$. From (3.3) we have

$$
s_0(g') = g' + (f_1 - f_2)\frac{\alpha_0}{f_0} - (\frac{\alpha_0}{f_0})^2 = g' - \alpha_0\frac{f_0'}{f_0^2}
\tag{3.14}
$$

by (1.11). Hence we have

$$
s_i(g') = g' - \alpha_i\frac{f_i'}{f_i^2} \quad (i = 0, 1, 2), \quad \pi(g') = g'.
\tag{3.15}
$$

In view of these, we define the action of $\widetilde{W}$ on g by

$$
s_i(g) = g + \frac{\alpha_i}{f_i} \quad (i = 0, 1, 2), \quad \pi(g) = g.
\tag{3.16}
$$

One can check that (3.16) gives rise in fact to a representation of $\widetilde{W}$ on $\mathbb{C}(\alpha; f)(g)$. On the variables F_0', F_1', F_2', equation (3.5) together with (3.16) immediately implies

$$
s_i(F_j') = F_j' \quad (i \neq j), \quad \pi(F_j') = F_{j+1}' \quad (i, j = 0, 1, 2).
\tag{3.17}
$$

These formulas justify the definitions of (3.13) other than those for $s_i(\tau_i)$ $(i = 0, 1, 2)$. As to $s_0(\tau_0)$, we compute

$$
s_0(F_0') = F_0' + \frac{\alpha_0}{f_0} = F_0' + \frac{f_0'}{f_0} + f_1 - f_2 = -F_0' + F_1' + F_2' + \frac{f_0'}{f_0}.
\tag{3.18}
$$

This leads to the definition

$$
s_0(\tau_0) = \frac{\tau_1\tau_2}{\tau_0}f_0 = \frac{\tau_1\tau_2}{\tau_0}(\frac{\tau_1'}{\tau_1} - \frac{\tau_2'}{\tau_2} + x) = \frac{1}{\tau_0}(D_x + x)\,\tau_1 \cdot \tau_2.
\tag{3.19}
$$

One can check by straightforward computations that the definition (3.13) thus obtained defines an action of $\widetilde{W}$ on the differential field $\mathbb{C}(\alpha)(x, \tau_0, \tau_1, \tau_2, \tau_0', \tau_1', \tau_2')$ as a group of differential automorphisms. □

We remark that the Bäcklund transformations $s_i(\tau_i)$ of Theorem 3.2 possibly become zero for solutions reducible to Riccati equations, while they can be applied

$$
\begin{tikzcd}
s_1(\tau_1) \arrow[rr] & & \tau_2 \arrow[rr] \arrow[dl, "f_1"'] & & s_0(\tau_0) \arrow[dl] \\
& \tau_0 \arrow[ul] \arrow[rr, "f_2"'] & & \tau_1 \arrow[ul, "f_0"'] \arrow[dl] & \\
& & s_2(\tau_2) \arrow[ul] & &
\end{tikzcd}
$$

FIGURE 1. Six τ-functions

repeatedly as long as the τ-functions remain nonzero. If (τ_0, τ_1, τ_2) is a *generic* solution, we obtain the Bäcklund transformations $(w(\tau_0), w(\tau_1), w(\tau_2))$ for any $w \in \widetilde{W}$, by Theorem 3.2.

From the formula (3.19) in the proof of Theorem 3.2, we have

Corollary 3.3. *In terms of the τ-functions τ_0, τ_1, τ_2, the dependent variables f_0, f_1,f_2 of the fourth Painlevé equation* (1.11) *are expressed multiplicatively as follows:*

$$
f_0 = \frac{\tau_0\, s_0(\tau_0)}{\tau_1\, \tau_2}, \quad f_1 = \frac{\tau_1\, s_1(\tau_1)}{\tau_2\, \tau_0}, \quad f_2 = \frac{\tau_2\, s_2(\tau_2)}{\tau_0\, \tau_1}. \tag{3.20}
$$

The relation between the f-variables and the six τ-functions in Corollary 3.3 can be represented graphically as in Figure 1. Note also that (3.20) implies

$$
\tau_0^2 s_0(\tau_0) + \tau_1^2 s_1(\tau_1) + \tau_2^2 s_2(\tau_2) = 3x\, \tau_0\tau_1\tau_2. \tag{3.21}
$$

Example. As to the rational solution (2.9) the corresponding τ-functions and their adjacent Bäcklund transformations are given by

$$
(\tau_0, \tau_1, \tau_2) = (1, 1, 1), \quad (s_0(\tau_0), s_1(\tau_1), s_2(\tau_2)) = (x, x, x). \tag{3.22}
$$

As to the rational solution (2.10), we have

$$
\begin{aligned}
&(\tau_0, \tau_1, \tau_2) = (x^2 + 1, x, 1), \\
&(s_0(\tau_0), s_1(\tau_1), s_2(\tau_2)) = (1, x^2 - 1, x^4 + 2x^2 - 1).
\end{aligned} \tag{3.23}
$$

These are examples of *Okamoto polynomials* which will be discussed in the next section.

Another corollary of Theorem 3.2 is the Toda equations for our τ-functions.

Corollary 3.4. *The fourth Painlevé equation* (3.7) *for the triple of τ-functions τ_0, τ_1, τ_2 implies the following equation of Toda type:*

$$
(\log \tau_0)'' + x^2 + \frac{\alpha_1 - \alpha_2}{3} = \frac{s_1(\tau_1) s_2(\tau_2)}{\tau_0^2}, \tag{3.24}
$$

namely,

$$
(\frac{1}{2} D_x^2 + x^2 + \frac{\alpha_1 - \alpha_2}{3})\, \tau_0 \cdot \tau_0 = s_1(\tau_1) s_2(\tau_2). \tag{3.25}
$$

Proof. From Corollary 3.3, we have

$$f_1 f_2 = \frac{s_1(\tau_1)\, s_2(\tau_2)}{\tau_0^2}. \tag{3.26}$$

On the other hand, substitution of the formulas

$$F_2' - F_0' = f_1 - x, \quad F_0' - F_1' = f_2 - x, \quad F_1' - F_2' = 2x - f_1 - f_2 \tag{3.27}$$

into (3.12) with $i = 0$ gives

$$F_0'' + x^2 + \frac{\alpha_1 - \alpha_2}{3} = f_1 f_2. \tag{3.28}$$

Equating (3.26) and (3.28) we obtain the equation of Corollary as desired. □.

Remark. Our τ-functions are slightly different from those introduced by K. Okamoto [7]. In our formulation, the τ-function in the spirit of Okamoto, say τ^{ok}, can be defined through the integral of a "Hamiltonian" as follows:

$$H = \frac{1}{3}(f_0 f_1 f_2 + \alpha_1 f_2 - \alpha_2 f_1) = (\log \tau^{\mathrm{ok}})'. \tag{3.29}$$

Note that this implies $(\log \tau^{\mathrm{ok}})'' = H' = f_1 f_2$. Let us introduce the triple of τ-functions of Okamoto type by

$$(\log \tau_0^{\mathrm{ok}})' = H_0, \quad (\log \tau_1^{\mathrm{ok}})' = H_1, \quad (\log \tau_2^{\mathrm{ok}})' = H_2, \tag{3.30}$$

where we define $H_0 = H$, $H_1 = \pi(H)$, $H_2 = \pi^2(H)$ by rotation. This implies

$$f_0 = (\log \frac{\tau_1^{\mathrm{ok}}}{\tau_2^{\mathrm{ok}}})' + \alpha_0 x, \quad f_1 = (\log \frac{\tau_2^{\mathrm{ok}}}{\tau_0^{\mathrm{ok}}})' + \alpha_1 x, \quad f_2 = (\log \frac{\tau_0^{\mathrm{ok}}}{\tau_1^{\mathrm{ok}}})' + \alpha_2 x. \tag{3.31}$$

(Compare these formulas with our definition (3.1).) From (3.28) we also see that

$$\tau_0^{\mathrm{ok}} = e^{x^4/12 + (\alpha_1 - \alpha_2)x^2/6}\, \tau_0 \tag{3.32}$$

up to the multiplication by the exponential of a linear function in x.

4. Rational solutions

In this section, we give an explicit description of the rational solutions of the fourth Painlevé equation (1.11) in terms of Schur functions.

Before discussing the rational solutions, we introduce a family of τ-functions $(\tau_{m,n})_{m,n \in \mathbb{Z}}$ for the fourth Painlevé equation (3.7). A similar treatment of the lattice of τ-functions has been given by K. Okamoto [8]. We consider the elements

$$T_1 = \pi s_2 s_1, \quad T_2 = s_1 \pi s_2 \tag{4.1}$$

of the (extended) affine Weyl group $\widetilde{W}$. Note that these T_1 and T_2 represent the following parallel translations in the parameter space V respectively:

$$T_1.\mathbf{v} = \mathbf{v} + (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \quad T_2.\mathbf{v} = \mathbf{v} + (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}), \tag{4.2}$$

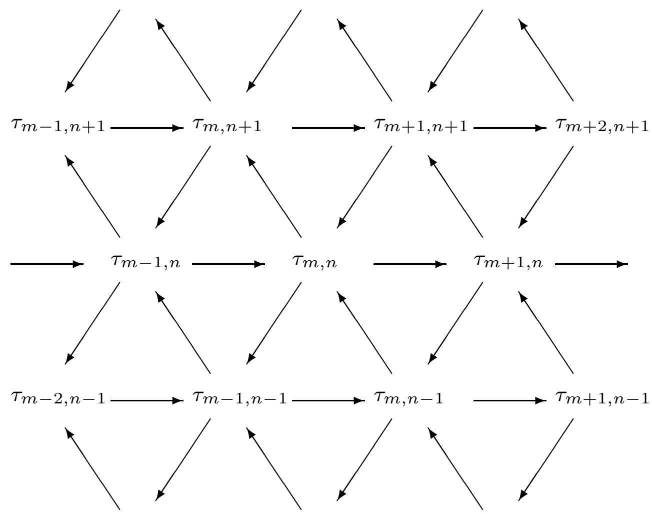


FIGURE 2. τ-Functions on the A_2-lattice

for $\mathbf{v} \in V$. For the triple of τ-functions (τ_0, τ_1, τ_2) of the fourth Painlevé equation (3.7), we introduce an infinite family of dependent variables $\tau_{m,n}$ $(m, n \in \mathbb{Z})$ as the Bäcklund transformations

$$\tau_{m,n} = T_1^m T_2^n(\tau_0) \quad (m, n \in \mathbb{Z}). \tag{4.3}$$

Note that

$$T_1(\tau_0) = \tau_1, \quad T_2(\tau_0) = s_1(\tau_1) \quad \text{and} \quad T_2T_1(\tau_0) = T_2(\tau_1) = \tau_2. \tag{4.4}$$

By these formulas, we have

$$\tau_{0,0} = \tau_0, \quad \tau_{1,0} = \tau_1, \quad \tau_{1,1} = \tau_2, \quad \tau_{0,1} = s_1(\tau_1). \tag{4.5}$$

The triple of τ-functions (τ_0, τ_1, τ_2) is transformed into $(\tau_{m,n}, \tau_{m+1,n}, \tau_{m+1,n+1})$ by $T_1^m T_2^n$, and into $(\tau_{m,n}, \tau_{m,n+1}, \tau_{m+1,n+1})$ by $T_1^m T_2^n s_1$, respectively. The following propositions are obtained immediately from the results of the previous section, by using the action of $\widetilde{W}$.

Proposition 4.1. (1) *For any $m, n \in \mathbb{Z}$, the triples*

$$(\tau_{m,n}, \tau_{m+1,n}, \tau_{m+1,n+1}) \quad \textit{and} \quad (\tau_{m,n}, \tau_{m,n+1}, \tau_{m+1,n+1}) \tag{4.6}$$

represent the Bäcklund transformations of (τ_0, τ_1, τ_2) for the parameters

$$\begin{aligned} &(\alpha_0 + 3m, \alpha_1 + 3(n - m), \alpha_2 - 3n) \quad \textit{and} \\ &(\alpha_0 + \alpha_1 + 3n, -\alpha_1 + 3(m - n), \alpha_1 + \alpha_2 - 3m), \end{aligned} \tag{4.7}$$

respectively.
(2) *The corresponding f-variables are given respectively by*

$$\left(\frac{\tau_{m,n}\tau_{m+2,n+1}}{\tau_{m+1,n}\tau_{m+1,n+1}}, \frac{\tau_{m+1,n}\tau_{m,n+1}}{\tau_{m+1,n+1}\tau_{m,n}}, \frac{\tau_{m+1,n+1}\tau_{m,n-1}}{\tau_{m,n}\tau_{m+1,n}}\right) \quad \textit{and} \tag{4.8}$$
$$\left(\frac{\tau_{m,n}\tau_{m+1,n+2}}{\tau_{m,n+1}\tau_{m+1,n+1}}, \frac{\tau_{m,n+1}\tau_{m+1,n}}{\tau_{m+1,n+1}\tau_{m,n}}, \frac{\tau_{m+1,n+1}\tau_{m-1,n}}{\tau_{m,n}\tau_{m,n+1}}\right).$$

Proposition 4.2. (1) *The family of τ-functions $\tau_{m,n}$ $(m,n\in\mathbb{Z})$ satisfies the following three types of bilinear equations:*

$$\begin{aligned}(D_x+x)\,\tau_{m,n}\cdot\tau_{m+1,n} &= \tau_{m,n-1}\tau_{m+1,n+1},\\ (D_x+x)\,\tau_{m,n}\cdot\tau_{m,n+1} &= \tau_{m+1,n+1}\tau_{m-1,n},\\ (D_x+x)\,\tau_{m,n}\cdot\tau_{m-1,n-1} &= \tau_{m-1,n}\tau_{m,n-1}.\end{aligned}\tag{4.9}$$

(2) *The family of τ-functions $\tau_{m,n}$ $(m,n\in\mathbb{Z})$ satisfies the following three types of Toda equations:*

$$\begin{aligned}&(\frac{1}{2}D_x^2+x^2-\frac{2\alpha_1+\alpha_2}{3}+2m-n)\,\tau_{m,n}\cdot\tau_{m,n}=\tau_{m+1,n}\tau_{m-1,n},\\ &(\frac{1}{2}D_x^2+x^2+\frac{\alpha_1-\alpha_2}{3}-m+2n)\,\tau_{m,n}\cdot\tau_{m,n}=\tau_{m,n+1}\tau_{m,n-1},\\ &(\frac{1}{2}D_x^2+x^2+\frac{\alpha_1+2\alpha_2}{3}-m-n)\,\tau_{m,n}\cdot\tau_{m,n}=\tau_{m-1,n-1}\tau_{m+1,n+1}.\end{aligned}\tag{4.10}$$

Remark. As we already remarked in the previous section, Bäcklund transformations for τ-functions possibly become singular, when applied to particular solutions which are reducible to Riccati equations. In such cases, we need to restrict the indices (m,n) for $\tau_{m,n}$ to a region of $\mathbb{Z}^2$ bounded by certain lines on which $\tau_{m,n}=0$.

All the rational solutions of (1.11) are obtained from

$$\begin{aligned}&\text{(A)}\qquad (\alpha_0,\alpha_1,\alpha_2;f_0,f_1,f_2)=(1,1,1;\,x,x,x),\quad\text{or}\\ &\text{(B)}\qquad (\alpha_0,\alpha_1,\alpha_2;f_0,f_1,f_2)=(3,0,0;3x,0,0).\end{aligned}\tag{4.11}$$

by Bäcklund transformations. We will determine the τ-functions $\tau_{m,n}(m,n\in\mathbb{Z})$ for these rational solutions.

In the case of Bäcklund transformations of the rational solution (A) of (4.11), the τ-functions $\tau_{m,n}$ $(m,n\in\mathbb{Z})$ turn out to be polynomials, which we call the *Okamoto polynomials*. We recall that the τ-functions for (A) are given by

$$(\alpha_0,\alpha_1,\alpha_2;\tau_0,\tau_1,\tau_2)=(1,1,1;1,1,1).\tag{4.12}$$

Theorem 4.3. *The τ-functions $\tau_{m,n}(x)$ for the solution* (4.12) *are polynomials in x. These polynomials $\tau_{m,n}(x)=Q_{m,n}(x)$ $(m,n\in\mathbb{Z})$ are characterized by the Toda*

equations

$$
(4.13)\qquad
\begin{aligned}
&(\frac{1}{2}D_x^2 + x^2 - 1 + 2m - n)\, Q_{m,n}\cdot Q_{m,n} = Q_{m+1,n}Q_{m-1,n},\\
&(\frac{1}{2}D_x^2 + x^2 - m + 2n)\, Q_{m,n}\cdot Q_{m,n} = Q_{m,n+1}Q_{m,n-1},\\
&(\frac{1}{2}D_x^2 + x^2 + 1 - m - n)\, Q_{m,n}\cdot Q_{m,n} = Q_{m-1,n-1}Q_{m+1,n+1}
\end{aligned}
$$

with initial condition

$$
(4.14)\qquad Q_{0,0} = Q_{1,0} = Q_{1,1} = 1, \quad Q_{2,1} = x.
$$

We remark that $Q_m(x) = Q_{m,0}(x)$ and $R_m(x) = Q_{m+1,1}(x)$ $(m \in \mathbb{Z})$ are the original Okamoto polynomials discussed in [1]. In fact, they are determined by the recurrence relations

$$
(4.15)\qquad (\frac{1}{2}D_x^2 + x^2 + 2m - 1)\, Q_m\cdot Q_m = Q_{m+1}Q_{m-1} \quad (m \in \mathbb{Z})
$$

with initial condition $Q_0 = Q_1 = 1$, and by

$$
(4.16)\qquad (\frac{1}{2}D_x^2 + x^2 + 2m)\, R_m\cdot R_m = R_{m+1}R_{m-1} \quad (m \in \mathbb{Z})
$$

with $R_0 = 1, R_1 = x$, respectively. The fact that $\tau_{m,n}(x)$ are polynomials will be proved in Section 5 in the course of the proof of Theorem 4.5 below. The other statements in Theorem 4.3 are consequences of Proposition 4.2.

The τ-functions for the rational solution (B) of (4.11) are given by

$$
(4.17)\qquad (\alpha_0, \alpha_1, \alpha_2; \tau_0, \tau_1, \tau_2) = (3, 0, 0; e^{-x^4/12}, e^{-x^4/12+x^2/2}, e^{-x^4/12-x^2/2}).
$$

Theorem 4.4. *The τ-functions $\tau_{m,n}(x)$ for the solution (4.17) are defined for $(m,n) \in \mathbb{Z}^2$ with $m \geq n \geq 0$. They can be written in the form*

$$
(4.18)\qquad \tau_{m,n}(x) = \exp\left(-\frac{x^4}{12} + \frac{m-2n}{2}x^2\right) H_{m-n,n} \quad (m \geq n \geq 0),
$$

for some polynomials $H_{m,n}(x)$. These polynomials $H_{m,n}(x)$ $(m, n \geq 0)$ are characterized by the Toda equations

$$
(4.19)\qquad
\begin{aligned}
&(\frac{1}{2}D_x^2 + 3m)\, H_{m,n}\cdot H_{m,n} = H_{m+1,n}H_{m-1,n},\\
&(\frac{1}{2}D_x^2 - 3n)\, H_{m,n}\cdot H_{m,n} = H_{m,n+1}H_{m,n-1},
\end{aligned}
$$

with initial condition

$$
(4.20)\qquad H_{0,0} = H_{1,0} = H_{0,1} = 1 \quad \textit{and} \quad H_{1,1} = 3x.
$$

We remark that $H_{m,1}(x)$ and $H_{1,m}$ ($m = 0, 1, 2, \dots$) coincide with the Hermite polynomials up to rescaling. We will call $H_{m,n}(x)$ ($m, n \geq 0$) the *generalized Hermite polynomials*. The fact that $\tau_{m,n}(x)$ are expressed as in (4.18) will be proved in Section 5 in the course of the proof of Theorem 4.6 below.

The Okamoto polynomials $Q_{m,n}(x)$ ($m, n \in \mathbb{Z}$) and the generalized Hermite polynomials $H_{m,n}(x)$ ($m, n \geq 0$) are in fact expressible in terms of Schur functions. We recall the definition of Schur functions in order to make this statement precise.

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ (or a *Young diagram*) is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and that $\lambda_i = 0$ for $i \gg 0$. The number of nonzero parts λ_i is called the *length* of λ and denoted by $l(\lambda)$. For each partition λ, we define the *Schur function* $S_\lambda(t) = S_\lambda(t_1, t_2, \dots)$ by

$$S_\lambda(t) = \det\left(p_{\lambda_i - i + j}(t)\right)_{1 \leq i,j \leq l(\lambda)}, \tag{4.21}$$

where $p_n(t)$ are the polynomials in t determined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{n=0}^{\infty} p_n(t) z^n. \tag{4.22}$$

(We set $p_n(t) = 0$ for $n < 0$.) Note that $p_n(t)$ can be defined equivalently by

$$p_n(t) = \sum\nolimits_{k_1 + 2k_2 + \cdots + nk_n = n} \frac{t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}}{k_1! k_2! \cdots k_n!}. \tag{4.23}$$

We say that a subset $M \subset \mathbb{Z}$ is a *Maya diagram* if

$$m \in M \quad (m \ll 0) \quad \text{and} \quad m \notin M \quad (m \gg 0). \tag{4.24}$$

To each Maya diagram $M = \{\dots, m_3, m_2, m_1\}$ $(\cdots < m_3 < m_2 < m_1)$, one can associate a unique partition $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$ for $i = 1, 2, \dots$. Note that all the Maya diagrams $M + k = \{\dots, m_2 + k, m_1 + k\}$ ($k \in \mathbb{Z}$) obtained from $M = \{\dots, m_3, m_2, m_1\}$ by shifting define the same partition by this correspondence. For each pair (m, n) of integers, we define the Maya diagram $M(m, n)$ as follows:

$$M(m, n) = 3D_m \cup (3D_n + 1) \cup (3D_0 + 2), \tag{4.25}$$

where

$$D_l = \{n \in \mathbb{Z} \mid n < l\} \quad (l \in \mathbb{Z}). \tag{4.26}$$

We denote by $\lambda(m, n)$ the partition corresponding to $M(m, n)$. Partitions of the form $\lambda(m, n)$ ($m, n \in \mathbb{Z}$) are called the 3-*reduced partitions*. We remark that a partition λ is 3-reduced if and only if λ has no hook with length of a multiple of 3. Also, Schur functions $S_{\lambda(m,n)}(t)$ for 3-reduced partitions are called *3-reduced Schur functions*. It is known that a Schur function $S_\lambda(t)$ is 3-reduced if and only if

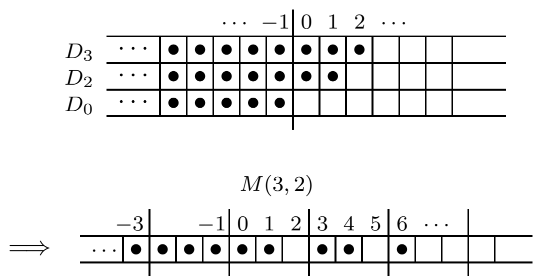
$$\partial_{t_{3n}} S_\lambda(t) = 0 \quad \text{for all} \quad n = 1, 2, \cdots. \tag{4.27}$$

Theorem 4.5. *Each Okamoto polynomial $Q_{m,n}(x)$ $(m, n \in \mathbb{Z})$ is a monic polynomial of degree $m^2 + n^2 - mn - m$ with integer coefficients. It is expressed by the 3-reduced Schur function as*

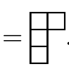
$$Q_{m,n}(x) = N_{m,n} S_{\lambda(m,n)}(x, \frac{1}{2}, 0, 0, \dots), \tag{4.28}$$

where $N_{m,n}$ is a positive integer determined by the hook-length formula.

Example. The Maya diagram $M(3,2)$ is obtained from D_3, D_2, D_0 as follows.



Hence we have $M(3,2) = \{\dots, -2, -1, 0, 1, 3, 4, 6\}$ and

$$\lambda(3,2) = (2,1,1) = \tag{4.29}$$  .

In this case, the Schur function $S_{(2,1,1)}(t)$ and the Okamoto polynomials $Q_{3,2}(x)$ are

$$\begin{aligned} S_{(2,1,1)}(t) &= \frac{1}{8}t_1^4 - \frac{1}{2}t_1^2 t_2 - \frac{1}{2}t_2^2 + t_4, \\ Q_{3,2}(x) &= x^4 - 2x^2 - 1, \end{aligned} \tag{4.30}$$

respectively. A typical sequence of 3-reduced partitions is given by

$$\lambda(m,0) = (2m-2, 2m-4, \dots, 2) \quad \text{for} \quad m > 0, \tag{4.31}$$

which corresponds to the Okamoto polynomials $Q_m(x)$ for $m > 0$. For other examples, see Figure 3 in the next section.

The generalized Hermite polynomials $H_{m,n}(x)$ are expressed by the Schur functions for rectangular Young diagrams $\lambda = (n^m) = (n, n, \dots, n, 0, 0, \dots)$.

Theorem 4.6. *Each generalized Hermite polynomial $H_{m,n}(x)$ $(m, n \geq 0)$ is a polynomial of degree mn with rational coefficients. It can be written as*

$$H_{m,n}(x) = C_{m,n} S_{(n^m)}(x, \frac{1}{6}, 0, 0, \dots), \tag{4.32}$$

where the normalization constant is given by

$$C_{m,n} = (-1)^{n(n-1)/2} 3^{(m+n)(m+n-1)/2} (m+n-1)^! \tag{4.33}$$

with $n^! = n!(n-1)!(n-2)! \cdots 2!1!$.

The relationship between the sequences $H_{1,n}(x)$, $H_{m,1}(x)$ and the ordinary Hermite polynomials $H_n(x)$ is obvious since

$$\sum_{n=0}^{\infty} S_{(n)}(x, \frac{1}{6}, 0, 0, \dots) z^n = \exp\left(xz + \frac{1}{6} z^2 \right), \tag{4.34}$$
$$\sum_{m=0}^{\infty} S_{(1^m)}(x, \frac{1}{6}, 0, 0, \dots) z^m = \exp\left(xz - \frac{1}{6} z^2 \right),$$

while the Hermite polynomials have the generating function

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n = \exp\left(2xz - z^2\right). \tag{4.35}$$

The proof of Theorems 4.5 and 4.6 will be given in the next section.

5. Proof of Theorems 4.5 and 4.6

The Hirota bilinear equations for our τ-functions (Theorem 3.1) arise naturally from the so-called *modified KP hierarchy* [2] by certain similarity reduction (see also [5], [10]). This fact is the key to the proof of Theorems 4.5 and 4.6.

Consider two functions $G_0(t_1, t_2)$ and $G_1(t_1, t_2)$ in the two variables (t_1, t_2), and suppose that they satisfy the following Hirota bilinear equation

$$(D_{t_1}^2 + D_{t_2})\, G_0(t_1, t_2) \cdot G_1(t_1, t_2) = 0. \tag{5.1}$$

Introducing the degrees of t_1, t_2 by $\deg t_1 = 1$ and $\deg t_2 = 2$, we assume that each G_i is homogeneous of degree $d_i \in \mathbb{C}$ for $i = 0, 1$:

$$(t_1 \partial_{t_1} + 2t_2 \partial_{t_2}) G_i(t_1, t_2) = d_i G_i(t_1, t_2). \tag{5.2}$$

Fixing a constant $k \in \mathbb{C}^{\times}$, we define the functions $\tau_0(x), \tau_1(x)$ in one variable by

$$\tau_i(x) = G_i(x, k) \quad (k \in \mathbb{C}^{\times};\ i = 0, 1). \tag{5.3}$$

Formally, each G_i is recovered by the formula

$$G_i(t_1, t_2) = \left(\frac{t_2}{k} \right)^{\frac{d_i}{2}} \tau_i \left(\frac{t_1}{\sqrt{t_2/k}} \right). \tag{5.4}$$

Then it is easy to check

Lemma 5.1. *Under the similarity condition* (5.2)*, the equation* (5.1) *for the pair* $G_0(t_1, t_2)$*,* $G_1(t_1, t_2)$ *is equivalent to the Hirota bilinear equation*

$$\left(2kD_x^2 - xD_x + d_0 - d_1\right)\ \tau_0(x) \cdot \tau_1(x) = 0, \tag{5.5}$$

for $\tau_0(x)$*,* $\tau_1(x)$*.*

From this lemma with $k = 1/2$, we immediately have

Proposition 5.2. *The fourth Painlevé equation* (3.7) *for the triple of τ-functions $\tau_0(x), \tau_1(x), \tau_2(x)$ is equivalent to the similarity reduction of the Hirota equations*

$$(D_{t_1}^2 + D_{t_2})\ G_i(t_1, t_2) \cdot G_{i+1}(t_1, t_2) = 0 \quad (i = 0, 1, 2) \tag{5.6}$$

for three functions $G_i(t_1, t_2)$ $(i = 0, 1, 2)$ in two variables. The similarity condition is given by

$$G_i(t_1, t_2) = (2t_2)^{d_i/2}\, \tau_i\left(\frac{t_1}{\sqrt{2t_2}}\right) \quad (i = 0, 1, 2), \tag{5.7}$$

and the parameters are related by

$$\begin{aligned} \alpha_0 &= 1 - 2d_0 + d_1 + d_2, \\ \alpha_1 &= 1 + d_0 - 2d_1 + d_2, \\ \alpha_2 &= 1 + d_0 + d_1 - 2d_2. \end{aligned} \tag{5.8}$$

Recall that the (first) *modified KP hierarchy* [2] is the following system of Hirota bilinear equations for a pair of τ-functions $\tau_0(t)$ and $\tau_1(t)$ in infinite time variables $t = (t_1, t_2, \dots)$:

$$\sum_{n=0}^{\infty} p_n(-2s)\, p_{n+2}(\widetilde{D}_t) \exp\left(\sum_{m=1}^{\infty} s_m D_{t_m}\right)\ \tau_0(t) \cdot \tau_1(t) = 0, \tag{5.9}$$

where $s = (s_1, s_2, \dots)$ are parameters and $\widetilde{D}_t = (D_{t_1}/1, D_{t_2}/2, \dots)$. The constant term of (5.9) with respect to s implies the bilinear equation

$$(D_{t_1}^2 + D_{t_2})\ \tau_0(t) \cdot \tau_1(t) = 0, \tag{5.10}$$

which is nothing but the equation (5.1) discussed above. For the proof of Theorems 4.5 and 4.6, we will recall the following fact about Schur functions from the theory of KP hierarchy. Let $X_m = X_m(t; \partial_t)$ $(m \in \mathbb{Z})$ be the *vertex operators* of the KP hierarchy defined by the generating function

$$X(z) = \sum_{m \in \mathbb{Z}} X_m z^m = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial_{t_k}\right). \tag{5.11}$$

Then we have

Lemma 5.3. *For any partition λ and $k \in \mathbb{Z}$, the pair $\tau_0(t) = S_\lambda(t)$ and $\tau_1(t) = X_k S_\lambda(t)$ solves the first modified KP hierarchy* (5.9). *In particular we have*

$$(D_{t_1}^2 + D_{t_2})\ \tau_0(t) \cdot \tau_1(t) = 0. \tag{5.12}$$

We will give a proof of this lemma in Appendix for completeness.

All the Schur functions $S_\lambda(t)$ are obtained from $S_\emptyset(t) = 1$ by applying vertex operators repeatedly:

$$S_\lambda(t) = X_{\lambda_1} \cdots X_{\lambda_n}.1, \tag{5.13}$$

for any partition $\lambda = (\lambda_1, \dots, \lambda_n, 0, \dots)$. The action of vertex operators on Schur functions can be computed by (5.13) together with the commutation relations

$$X_k X_l = -X_{l-1} X_{k+1}, \quad X_k.1 = 0 \ \ (k < 0), \quad X_0.1 = 1, \tag{5.14}$$

where $k, l \in \mathbb{Z}$. (See Appendix.) A more systematic way is to use Maya diagrams. For a given Maya diagram M, let λ be the corresponding partition and suppose that $l(\lambda) \le n$. Then we have

$$X_k.S_\lambda(t) = \begin{cases} \pm S_\mu(t) & \text{if} \quad k+n \notin M, \\ 0 & \text{if} \quad k+n \in M, \end{cases} \tag{5.15}$$

for each $k \in \mathbb{Z}$. Here μ stands for the partition corresponding to the Maya diagram $M \cup \{k+n\}$. The sign in this formula is determined by the parity of the number of integers $m \in M$ such that $m > k+n$.

Proof of Theorem 4.5 *for Okamoto polynomials.* By using the formula (5.15), one can compute how the 3-reduced Schur functions are transformed by vertex operators.

Lemma 5.4. *As to the action of the vertex operators, we have the following two types of cyclic relations among 3-reduced Schur functions*

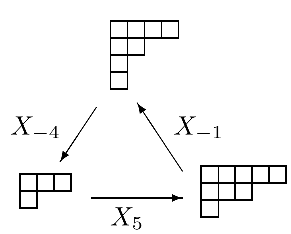
$$\begin{aligned} X_{2m-n}\,.S_{\lambda(m,n)}(t) &= \pm S_{\lambda(m+1,n)}(t), \\ X_{2n-m}\,.S_{\lambda(m+1,n)}(t) &= \pm S_{\lambda(m+1,n+1)}(t), \\ X_{-m-n}\,.S_{\lambda(m+1,n+1)}(t) &= \pm S_{\lambda(m,n)}(t), \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} X_{2n-m+1}\,.S_{\lambda(m,n)}(t) &= \pm S_{\lambda(m,n+1)}(t), \\ X_{2m-n-1}\,.S_{\lambda(m,n+1)}(t) &= \pm S_{\lambda(m+1,n+1)}(t), \\ X_{-m-n}\,.S_{\lambda(m+1,n+1)}(t) &= \pm S_{\lambda(m,n)}(t), \end{aligned} \tag{5.17}$$

for any $m, n \in \mathbb{Z}$.

Example. Consider the 3-reduced partitions $\lambda(3,1) = (3,1)$, $\lambda(4,1) = (5,3,1)$ and $\lambda(4,2) = (4,2,1,1)$. For this triple of partitions, we have a 'cycle' of 3-reduced Schur functions



which is an example of (5.16) for $(m,n) = (3,1)$. Notice also that the index of each vertex operator X_k represents the difference of degrees of Schur functions.

From Lemma 5.4 together with Lemma 5.3, we obtain two types of triples of 3-reduced Schur functions satisfying the bilinear equations of Proposition 5.2. Note that the 3-reduced Schur function $S_{\lambda(m,n)}$ is homogeneous of degree

$$d_{m,n} = |\lambda(m,n)| = m^2 + n^2 - mn - m \tag{5.18}$$

with respect to the degree defined by $\deg t_i = i$ $(i = 1, 2, \dots)$. Set

$$s_{m,n}(x) = S_{\lambda(m,n)}(x, \frac{1}{2}, 0, 0, \dots) \tag{5.19}$$

for any $m, n \in \mathbb{Z}$. Then, by combining Lemma 5.4 and Proposition 5.2, we have

Proposition 5.5. (1) *For any $m, n \in \mathbb{Z}$, the triple*

$$(s_{m,n}(x), s_{m+1,n}(x), s_{m+1,n+1}(x)) \tag{5.20}$$

solves the fourth Painlevé equation (3.7) *with the parameters*

$$(\alpha_0, \alpha_1, \alpha_2) = (3m + 1, 3(n - m) + 1, -3n + 1). \tag{5.21}$$

(2) *For any $m, n \in \mathbb{Z}$, the triple*

$$(s_{m,n}(x), s_{m,n+1}(x), s_{m+1,n+1}(x)) \tag{5.22}$$

solves the fourth Painlevé equation (3.7) *with parameters*

$$(\alpha_0, \alpha_1, \alpha_2) = (3n + 2, 3(m - n) - 1, -3m + 2). \tag{5.23}$$

We remark that, in the coordinates (v_1, v_2, v_3) of the parameter space V as in (1.10), the triples of τ-functions in this proposition give rise to solutions with parameters

$$(v_1, v_2, v_3) = (\frac{1}{3}, 0, -\frac{1}{3}) - m(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) - n(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}) \tag{5.24}$$

and

$$(v_1, v_2, v_3) = (0, \frac{1}{3}, -\frac{1}{3}) - m(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}) - n(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), \tag{5.25}$$

respectively.

It is clear that each triple of τ-functions of Proposition 5.5 defines a rational solution of (1.11) in f-variables. For each $(\alpha_0, \alpha_1, \alpha_2)$ of this proposition, the fourth Painlevé equation (1.11) has a unique rational solution by [4]. Hence we conclude that each $s_{m,n}(x)$ is a constant multiple of the τ-function $\tau_{m,n}(x)$ for the solution (A) of (4.11). This shows that $\tau_{m,n}(x)$ are in fact polynomials in x. The assertion that the Okamoto polynomials $\tau_{m,n}(x) = Q_{m,n}(x)$ are monic polynomials with integer coefficients follows either from the Bäcklund transformations or from the Toda equations of Proposition 4.2.

Proof of Theorem 4.6 *for generalized Hermite polynomials.* By the formula (5.15), we have

Lemma 5.6. *Under the action of vertex operators, we have the following relations among Schur functions for rectangular Young diagrams:*

$$\begin{aligned} X_{-m}.S_{((n+1)^m)}(t) &= (-1)^m S_{(n^m)}(t), \\ X_{n-m}.S_{((n+1)^m)}(t) &= (-1)^m S_{(n^{(m+1)})}(t), \\ X_n.S_{(n^m)}(t) &= S_{(n^{(m+1)})}(t), \end{aligned} \tag{5.26}$$

for $m, n \geq 0$.

For each $m, n \geq 0$, let

$$h_{m,n}(x) = S_{(n^m)}(x, \frac{1}{6}, 0, 0, \dots) \tag{5.27}$$

be the specialization of the Schur function $S_\lambda(t) = S_{(n^m)}(t)$ associated with rectangular Young diagram $\lambda = (n^m)$. Then by Lemma 5.1 (with $k = 1/6$), we have

Lemma 5.7.

$$\begin{aligned} &\left(D_x^2 - 3xD_x + 3m\right) h_{m,n+1} \cdot h_{m,n} = 0, \\ &\left(D_x^2 - 3xD_x + 3(m-n)\right) h_{m,n+1} \cdot h_{m+1,n} = 0, \\ &\left(D_x^2 - 3xD_x - 3n\right) h_{m,n} \cdot h_{m+1,n} = 0. \end{aligned} \tag{5.28}$$

These relations do not fit directly for the triple of τ-functions as in (3.7) since they do not make a 'cycle'. This problem can be repaired however by changing the normalization of $h_{m,n}$ as follows:

$$u_{m,n}(x) = \exp\left(-\frac{x^4}{12} + \frac{m-n}{2}x^2\right) h_{m,n}(x). \tag{5.29}$$

Then we have

Proposition 5.8.

$$\begin{aligned} &\left(D_x^2 - xD_x + m + 2n + 1\right) u_{m,n+1} \cdot u_{m,n} = 0, \\ &\left(D_x^2 - xD_x + m - n\right) u_{m+1,n} \cdot u_{m,n+1} = 0, \\ &\left(D_x^2 - xD_x - 2m - n - 1\right) u_{m,n} \cdot u_{m+1,n} = 0. \end{aligned} \tag{5.30}$$

Namely, the triple

$$(\tau_0, \tau_1, \tau_2) = (u_{m,n}, u_{m+1,n}, u_{m,n+1}) \tag{5.31}$$

solves the fourth Painlevé equation (3.7) *with parameters*

$$(\alpha_0, \alpha_1, \alpha_2) = (3(m+n+1), -3m, -3n). \tag{5.32}$$

Proof. Note that the Hirota bilinear equations have the following formulas of Leibniz type:

$$\begin{aligned} D_x(g_1u_1 \cdot g_2u_2) &= D_x(g_1 \cdot g_2)u_1u_2 + g_1g_2D_x(u_1 \cdot u_2), \\ D_x^2(g_1u_1 \cdot g_2u_2) &= D_x^2(g_1 \cdot g_2)u_1u_2 + 2D_x(g_1 \cdot g_2)D_x(u_1 \cdot u_2) + g_1g_2D_x^2(u_1 \cdot u_2). \end{aligned}$$

Applying these to $g_i = \exp(x^4/12 + a_i x^2/2)$ $(i = 1, 2)$, we have

$$
\begin{aligned}
&(D_x^2 - 3xD_x + \beta)(g_1u_1 \cdot g_2u_2) \\
= &\, g_1g_2\big\{D_x^2 + (2a_{12} - 3)xD_x + (2 - 3a_{12} + a_{12}^2)x^2 + (a_1 + a_2) + \beta\big\}u_1 \cdot u_2,
\end{aligned}
$$

where $a_{12} = a_1 - a_2$. The equations (5.30) can be checked easily by using this formula. □

We remark that, in the coordinates (v_1, v_2, v_3) of V, the solutions of Proposition 5.8 have the parameters

$$
(5.33)\quad (v_1, v_2, v_3) = -m(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) + n(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}) \quad (m, n = 0, 1, 2, \ldots).
$$

As in the case of Okamoto polynomials, we see that each $\tau_{m,n}$ $(m \geq n \geq 0)$ for the solution (B) of (4.11) is a constant multiple of $u_{m-n,n}$ by comparing the parameters. Hence we see that $\tau_{m,n}(x)$ has the expression of (4.18). The only problem remaining is to fix the constant factors. The leading coefficient of the polynomial $H_{m,n}(x)$ is given by

$$
(5.34)\qquad (-1)^{n(n-1)/2}(m-1)^!(n-1)^!3^{(m+n)(m+n-1)/2},
$$

which can be determined inductively by the Toda equations of Theorem 4.4. On the other hand, the leading coefficient of $h_{m,n}(x)$ is determined as

$$
(5.35)\qquad (m-1)^!(n-1)^!/(m+n-1)^!,
$$

by the hook-length formula. Hence we have Theorem 4.6.

We show in Figures 3 and 4 below how the τ-functions for rational solutions are arranged on the A_2-lattice. Also, we include some examples of Okamoto polynomials and generalized Hermite polynomials of small degrees.

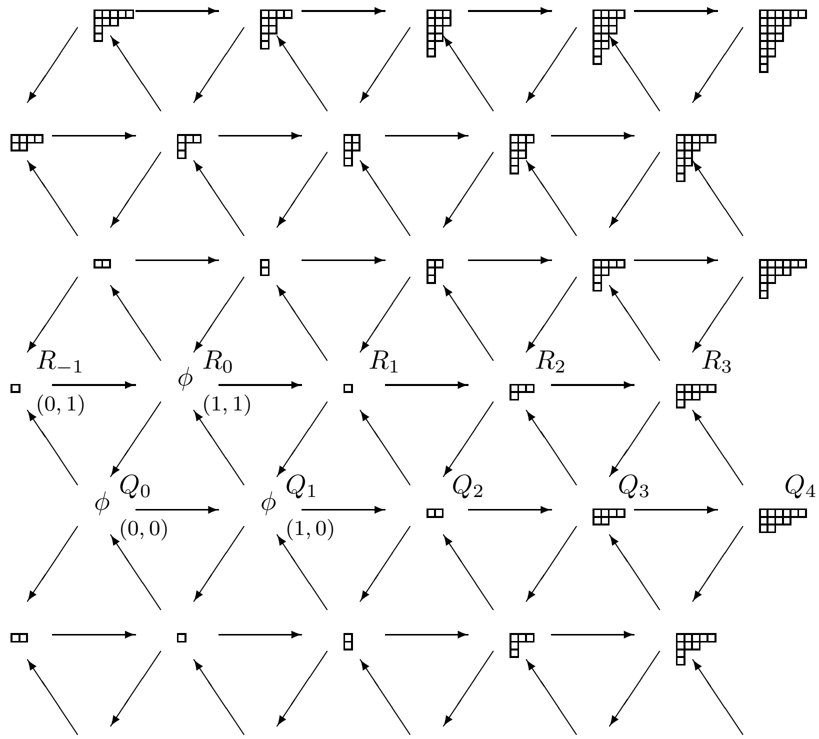


FIGURE 3. Okamoto polynomials on the A_2-lattice

Okamoto polynomials. In the following, we use the notation $Q_\lambda(x) = Q_{m,n}(x)$ for the Okamoto polynomial associated with the 3-reduced partition $\lambda = \lambda(m,n)$. We give below some examples of Okamoto polynomials $Q_\lambda(x)$.

$$
\begin{aligned}
&Q_{(0)} = Q_1 = 1, \quad Q_{(1)} = R_1 = x, \quad Q_{(2)} = Q_2 = 1 + x^2,\\
&Q_{(1,1)} = -1 + x^2, \quad Q_{(3,1)} = R_2 = -1 + 2x^2 + x^4,\\
&Q_{(2,1,1)} = -1 - 2x^2 + x^4, \quad Q_{(3,1,1)} = -5x + x^5,\\
&Q_{(4,2)} = Q_3 = 5 + 5x^2 + 5x^4 + x^6, \quad Q_{(2,2,1,1)} = -5 + 5x^2 - 5x^4 + x^6,\\
&Q_{(4,2,1,1)} = -7 - 14x^4 + x^8, \quad Q_{(5,3,1)} = R_3 = -35x + 14x^5 + 8x^7 + x^9,\\
&Q_{(5,3,1,1)} = 25 - 75x^2 - 50x^4 - 10x^6 + 5x^8 + x^{10},\\
&Q_{(6,4,2)} = Q_4 = 175 + 350x^2 + 175x^4 + 140x^6 + 65x^8 + 14x^{10} + x^{12},\\
&Q_{(7,5,3,1)} = R_4 = 1225 - 4900x^2 - 4900x^4 - 980x^6\\
&\quad + 350x^8 + 420x^{10} + 140x^{12} + 20x^{14} + x^{16},\\
&Q_{(8,6,4,2)} = Q_5 = 67375 + 134750x^2 + 202125x^4 + 107800x^6\\
&\quad + 42350x^8 + 20020x^{10} + 8050x^{12} + 2200x^{14} + 355x^{16} + 30x^{18} + x^{20}.
\end{aligned}
$$

Note that the original Okamoto polynomials are given by

$$
\begin{aligned}
&Q_n = Q_{(2n-2,2n-4,\dots,4,2)} \quad (n>0), \quad Q_{-n} = Q_{(n,n,\dots,2,2,1,1)} \quad (n \ge 0),\\
&R_n = Q_{(2n-1,2n-3,\dots,3,1)} \quad (n>0), \quad R_{-n} = Q_{(n,n+1,n+1,\dots,1,1)} \quad (n \ge 0).
\end{aligned}
$$

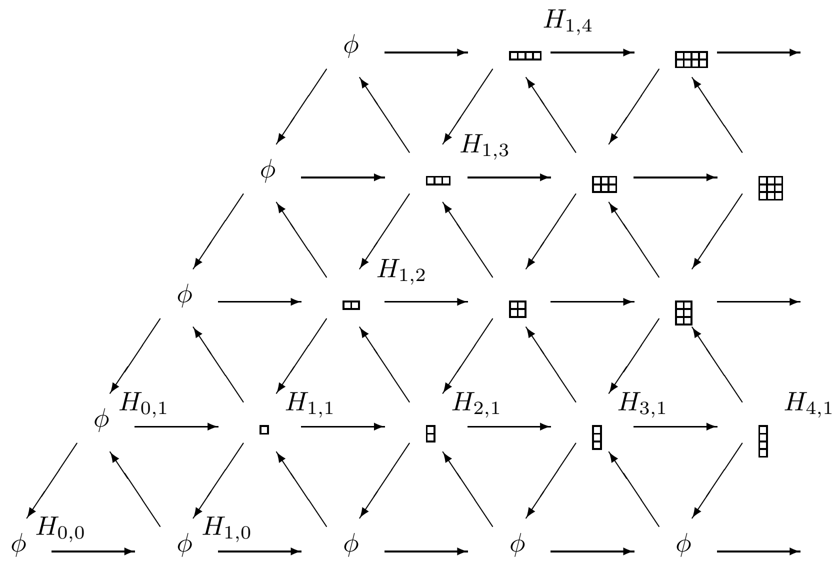


FIGURE 4. Generalized Hermite polynomials on the A_2-lattice

Generalized Hermite polynomials. The polynomials $H_{n,1}(x)$ and $H_{1,n}(x)$ coincide with the ordinary Hermite polynomials up to rescaling.

$$H_{0,0} = 1, \quad H_{1,0} = 1, \quad H_{2,0} = 3, \quad H_{3,0} = 2^1 3^3, \quad H_{4,0} = 2^2 3^7,$$

$$H_{0,1} = 1, \quad H_{1,1} = 3x, \quad H_{2,1} = 3^3(-\frac{1}{3} + x^2),$$

$$H_{3,1} = 2^1 3^6(-x + x^3), \quad H_{4,1} = 2^2 3^{11}(\frac{1}{3} - 2x^2 + x^4),$$

$$H_{0,2} = -3, \quad H_{1,2} = -3^3(\frac{1}{3} + x^2), \quad H_{2,2} = -3^6(\frac{1}{3} + x^4),$$

$$H_{3,2} = -2^1 3^{10}(\frac{1}{3} + x^2 - x^4 + x^6), \quad H_{4,2} = -2^2 3^{16}(\frac{5}{9} + \frac{10}{3}x^4 - \frac{8}{3}x^6 + x^8),$$

$$H_{0,3} = -2^1 3^3, \quad H_{1,3} = -2^1 3^6(x + x^3),$$

$$H_{2,3} = -2^1 3^{10}(-\frac{1}{3} + x^2 + x^4 + x^6), \quad H_{3,3} = -2^2 3^{15}(\frac{-5}{3}x + 2x^5 + x^9),$$

$$H_{4,3} = -2^3 3^{22}(\frac{25}{27} - \frac{50}{9}x^2 - \frac{25}{9}x^4 - \frac{20}{9}x^6 + 5x^8 - 2x^{10} + x^{12}),$$

$$H_{0,4} = 2^2 3^7, \quad H_{1,4} = 2^2 3^{11}(\frac{1}{3} + 2x^2 + x^4),$$

$$H_{2,4} = 2^2 3^{16}(\frac{5}{9} + \frac{10}{3}x^4 + \frac{8}{3}x^6 + x^8),$$

$$H_{3,4} = 2^3 3^{22}(\frac{25}{27} + \frac{50}{9}x^2 - \frac{25}{9}x^4 + \frac{20}{9}x^6 + 5x^8 + 2x^{10} + x^{12}),$$

$$H_{4,4} = 2^4 3^{30}(\frac{875}{243} + \frac{3500}{81}x^4 - \frac{50}{9}x^8 + \frac{20}{3}x^{12} + x^{16}).$$

A. Appendix

In this Appendix, we give a brief summary of relevant facts on Schur functions and their relation to KP-hierarchy for the sake of reference.

A.1. Schur functions. A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and that $\lambda_i = 0$ for $i \gg 0$. The number of nonzero λ_i is called the length of λ and denoted by $l(\lambda)$. For each partition λ, the Schur function $S_\lambda(t) = S_\lambda(t_1, t_2, \dots)$ is defined as follows:

$$S_\lambda(t) = \det \left(p_{\lambda_i - i + j}(t)\right)_{1 \leq i,j \leq l(\lambda)}, \tag{A.1}$$

where $p_n(t)$ are the polynomials defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{n=0}^{\infty} p_n(t) z^n. \tag{A.2}$$

Usually, the Schur functions are defined as the following character polynomials of the general linear group $GL(N, \mathbb{C})$ $(N \geq l(\lambda))$:

$$s_\lambda(x_1, \dots, x_N) = \frac{\det(x_j^{\lambda_i + \delta_i})}{\det(x_j^{\delta_i})}, \tag{A.3}$$

where $\delta_i = N - i$ $(i = 1, \dots, N)$. The polynomials $S_\lambda(t)$ and $s_\lambda(x)$ are related by $S_\lambda(t) = s_\lambda(x)$, where $t_k = \sum_{i=1}^n (x_i^k)/k$. In this context, the formula (A.1) above is the Jacobi-Trudi formula representing $s_\lambda(x)$ in terms of complete homogeneous symmetric functions.

The coefficients of $S_\lambda(t)$ with respect to the t-variables are related with irreducible character π_λ of the symmetric group $\mathfrak{S}_n$ of degree $n = |\lambda| = \sum_i \lambda_i$ as follows:

$$S_\lambda(t) = \sum_{m_1, m_2, \dots \geq 0} \pi_\lambda(1^{m_1} 2^{m_2} \cdots) \frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots, \tag{A.4}$$

where $\pi_\lambda(1^{m_1} 2^{m_2} \cdots)$ is the character value on the conjugate class of cycle type $(1^{m_1} 2^{m_2} \cdots)$. In particular, the coefficient of t_1^n is given by the hook-length formula

$$\frac{\pi_\lambda(1^n)}{n!} = \prod_{s \in \lambda} \frac{1}{h(s)}, \tag{A.5}$$

where $h(s) = \lambda_i + \lambda'_j - i - j + 1$, λ' being the conjugate partition, denotes the hook-length of λ at $s = (i, j)$.

A.2. (Modified) KP hierarchy. In the following, we use the notation

$$\xi(z, t) = \sum_{n=1}^{\infty} t_n z^n, \quad \xi(z^{-1}, \widetilde{\partial}_t) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{t_n}. \tag{A.6}$$

Consider the operators $V_k = V_k(z, t)$ $(k \in \mathbb{Z})$ defined by

$$V_k = e^{k\xi(z,t)} e^{-k\xi(z^{-1}, \widetilde{\partial}_t)}. \tag{A.7}$$

For each $m \in \mathbb{Z}$, we define the operators X_m and X_m^* as the coefficient of z^m in V_1 and V_{-1}, respectively:

$$\begin{aligned} V_1(z,t) =& X(z,t) = \sum_{m\in\mathbb{Z}} X_m z^m, \\ V_{-1}(z,t) =& X^*(z,t) = \sum_{m\in\mathbb{Z}} X_m^* z^m. \end{aligned} \tag{A.8}$$

By using the formula

$$V_k(z,t)V_l(w,t) = \left(1 - \frac{w}{z}\right)^{kl} e^{k\xi(z,t)+l\xi(w,t)} e^{-k\xi(z^{-1},\widetilde{\partial}_t)-l\xi(w^{-1},\widetilde{\partial}_t)}, \tag{A.9}$$

we obtain

Lemma A.1. *The vertex operators X_m and X_m^* ($m \in \mathbb{Z}$) satisfy the following anti-commutation relations:*

$$\begin{aligned} X_m X_n + X_{n-1}X_{m+1} = 0, \\ X_m^* X_n^* + X_{n-1}^* X_{m+1}^* = 0, \\ X_m X_n^* + X_{n+1}^* X_{m-1} = \delta_{m+n,0}. \end{aligned} \tag{A.10}$$

Proposition A.2. *For any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of length $l(\lambda) \le l$, we have*

$$S_\lambda(t) = X_{\lambda_1} \cdots X_{\lambda_l}.1. \tag{A.11}$$

Proof. By using (A.9), we have

$$X(z_1,t)\cdots X(z_l,t).1 = \prod_{1\le i<j\le l} \left(1 - \frac{z_j}{z_i}\right) \prod_{i=1}^{l} \exp\left(\sum_{n=1}^{\infty} t_n z_i^n\right). \tag{A.12}$$

By taking the coefficient of $z^\lambda = z_1^{\lambda_1}\cdots z_l^{\lambda_l}$ of this expression, we obtain the formula (A.1). $\square$

The KP hierarchy is a system of nonlinear partial differential equations for an unknown function $\tau(t) = \tau(t_1, t_2, \dots)$ including the Hirota bilinear equation

$$(D_{t_1}^4 - 4D_{t_1}D_{t_3} + 3D_{t_2}^2)\tau(t)\cdot\tau(t) = 0. \tag{A.13}$$

The whole system of the KP hierarchy is represented by the following bilinear relation:

$$\oint \frac{dz}{2\pi i} X^*(z,t')\tau(t')X(z,t)\tau(t) = 0. \tag{A.14}$$

Proposition A.3. *For any partition λ, the Schur function $S_\lambda(t)$ is a solution of the KP hierarchy.*

Proof. Note first that the bilinear equation (A.14) can be rewritten as follows:

$$\left(\sum_{m+n=-1} X_m^* \otimes X_n\right)\tau\otimes\tau = 0. \tag{A.15}$$

Here $\tau \otimes \tau = \tau(t')\tau(t)$ is regarded as an element of $\mathbb{C}[[t']] \otimes \mathbb{C}[[t]]$. By the anti-commutation relation (A.10), one has

$$\left(\sum_{m+n=-1} X_m^* \otimes X_n\right) X_k \otimes X_k = X_{k+1} \otimes X_{k-1} \left(\sum_{m+n=-1} X_m^* \otimes X_n\right) - 1 \otimes X_{k-1}X_k, \tag{A.16}$$

and the last term $X_{k-1}X_k$ vanishes. Hence, by applying the operator $X_{k+1} \otimes X_{k-1}$ to (A.15), it follows that $X_k\tau(t)$ is also a solution of the KP hierarchy. Starting from the solution $\tau(t) = 1$, we see that all the Schur functions are solutions of KP hierarchy by Proposition A.2 $\square$

Proposition A.4. *Let $\tau_0(t) = \tau(t)$ be any solution of the KP hierarchy, and put*

$$\tau_1(t) = X(w,t)\tau(t). \tag{A.17}$$

Then, we have

$$\oint \frac{dz}{2\pi i}\, z\, X^*(z,t')\tau_0(t')\ X(z,t)\tau_1(t) = 0. \tag{A.18}$$

Proof. Apply $X(w,t)$ to the second factor $X(z,t)\tau(t)$ of the bilinear relation (A.14). Then one obtains (A.18) by using the relation

$$X(w,t)X(z,t)\tau(t) = -\frac{z}{w}X(z,t)X(w,t)\tau(t) = -\frac{z}{w}X(z,t)\tau_1(t)$$

as desired. $\square$

The formula (A.18) is the bilinear relation of the first modified KP hierarchy.

By the change of variables $t \to t - s$ and $t' \to t + s$, the relations (A.14) and (A.18) can be rewritten into the following systems of Hirota bilinear equations

$$\sum_{n=0}^{\infty} p_n(-2s)p_{n+1}(\widetilde{D}_t) \exp\left(\sum_{m=1}^{\infty} s_m D_{t_m}\right) \tau(t) \cdot \tau(t) = 0, \tag{A.19}$$

$$\sum_{n=0}^{\infty} p_n(-2s)p_{n+2}(\widetilde{D}_t) \exp\left(\sum_{m=1}^{\infty} s_m D_{t_m}\right) \tau_0(t) \cdot \tau_1(t) = 0, \tag{A.20}$$

where $\widetilde{D}_{t_n} = D_{t_n}/n$. These are the Hirota bilinear equations for the τ-functions of the KP hierarchy and the first modified KP hierarchy, respectively.

References

1. S. Fukutani, K. Okamoto and H. Umemura, Special polynomials associated with the rational solutions and the Hirota bilinear relations of the 2nd and the 4th Painlevé equations, preprint.
2. M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. RIMS, Kyoto Univ. **19**(1983),943–1001.
3. M. Kac and P. van Moerbeke, On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices, Advances Math. **16** (1975), 160–169.
4. Y. Murata, Rational solutions of the second and the fourth equations of Painlevé, Funkcialaj Ekvacioj **28**(1985), 1–32.
5. T. Nishitani and M. Tajiri, On similarity solutions of Boussinesq equation, Phys. Lett. **89A** (1982), 379–380.
6. M. Noumi and K. Okamoto, Irreducibility of the second and the fourth Painlevé equations, Funkcialaj Ekvacioj **40** (1997), 139–163.
7. K. Okamoto, Studies on Painlevé equations III, Second and fourth Painlevé equations P_{II} and P_{IV}, Math. Ann. **275** (1986), 221–255.
8. K. Okamoto, Algebraic relations among six adjacent τ-functions related to the fourth Painlevé system, Kyushu J. Math. **50** (1996), 513–532.
9. H. Umemura, On the irreducibility of the first differential equations of Painlevé, in "Algebraic Geometry and Commutative Algebra", in honor of Masayoshi Nagata, 771–789, 1987.
10. G. R. W. Quispel, F. W. Nijhoff and H. W. Capel, Linearization of the Boussinesq equation and the modified Boussinesq equation, Phys. Lett. **91A** (1982), 143–145.

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